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On surgery along Brunnian links in 3–manifolds

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We consider surgery moves along $(n+1)$–component Brunnian links in compact connected oriented 3–manifolds, where the framing of the components is in $\{\frac{1}{k}\,;k\in\mathbf{Z}\}$. We show that no finite type invariant of degree $<2n-2$ can detect such a surgery move. The case of two link-homotopic Brunnian links is also considered. We relate finite type invariants of integral homology spheres obtained by such operations to Goussarov–Vassiliev invariants of Brunnian links.

57N10; 57M27

1 Introduction

In [19], Ohtsuki introduced the notion of finite type invariants of integral homology spheres as an attempt to unify the topological invariants of these objects, in the same way as Goussarov–Vassiliev invariants provide a unified point of view on invariants of knots and links. This theory was later generalized to all oriented 3–manifolds by Cochran and Melvin [2].

Goussarov and Habiro developed independently another finite type invariants theory for compact connected oriented 3–manifolds, which essentially coincides with the Ohtsuki theory in the case of integral homology spheres [4, 7, 12]. This theory comes equipped with a new and powerful tool called *calculus of clasper*, which uses embedded graphs carrying some surgery instruction. Surgery moves along claspers define a family of (finer and finer) equivalence relations among 3–manifolds, called *Y_k–equivalence*, which gives a good idea of the information contained by finite type invariants: two compact connected oriented 3–manifolds are not distinguished by invariants of degree $<k$ if they are Y_k–equivalent [8, 12]. These two conditions become equivalent when dealing with integral homology spheres.

Recall that a link L is *Brunnian* if any proper sublink of L is trivial. In some sense, an n–component Brunnian link is a 'pure n–component linking'. In this paper we consider those compact connected oriented 3–manifolds which are obtained by surgery along a Brunnian link. For a fixed number of components, we study which finite type invariants (ie of which degree) can vary under such an operation.

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Let $m=(m_1,...,m_n)\in\mathbf{Z}^n$ be a collection of n integers. Given a null-homologous, ordered n–component link L in a compact connected oriented 3–manifold M, denote by (L,m) the link L with framing $\frac{1}{m_i}$ on the i^{th} component ; $1\le i\le n$. We denote by $M_{(L,m)}$ the 3–manifold obtained from M by surgery along the framed link (L,m). We say that $M_{(L,m)}$ is obtained from M by $\frac{1}{m}$*–surgery* along the link L.

Theorem 1.1 *Let $n\ge 2$ and $m\in\mathbf{Z}^{n+1}$. Let L be an $(n+1)$–component Brunnian link in a compact, connected, oriented 3–manifold M.*

For $n=2$, $M_{(L,m)}$ and M are Y_1–equivalent.

For $n\ge 3$, $M_{(L,m)}$ and M are Y_{2n-2}–equivalent. Consequently, they cannot be distinguished by any finite type invariant of degree $<2n-2$.

Note that, for any Brunnian link L in M, we have $M_{(L,m)}\cong M$ if $m_i=0$ for some $1\le i\le n+1$. In this case, the statement is thus vacuous.

Two links are *link-homotopic* if they are related by a sequence of isotopies and self-crossing changes, ie, crossing changes involving two strands of the same component. We obtain the following.

Theorem 1.2 *Let $n\ge 2$ and $m\in\mathbf{Z}^{n+1}$. Let L and L' be two link-homotopic $(n+1)$–component Brunnian links in a compact, connected, oriented 3–manifold M. Then $M_{(L,m)}$ and $M_{(L',m)}$ are Y_{2n-1}–equivalent. Consequently, they cannot be distinguished by any finite type invariant of degree $<2n-1$.*

Actually, for integral homology spheres, the theorem is still true when "$2n-1$" is replaced by "$2n$". (It follows from the last observation of Section 3.7.)

In the latter part of the paper, we study the relation between the above results and Goussarov–Vassiliev invariants of Brunnian links.

Let $\mathbf{Z}\mathcal{L}(n)$ be the free $\mathbf{Z}$–module generated by the set of isotopy classes of n–component links in S^3. The theory of Goussarov–Vassiliev invariants of links involves a descending filtration

$$\mathbf{Z}\mathcal{L}(n)=J_0(n)\supset J_1(n)\supset J_2(n)\supset ...$$

called *Goussarov–Vassiliev filtration* (see Section 5.2). In a previous paper, Habiro and the author introduced the so-called *Brunnian part* $\mathrm{Br}(\bar{J}_{2n}(n+1))$ of $J_{2n}(n+1)/J_{2n+1}(n+1)$, which is defined as the $\mathbf{Z}$–submodule generated by elements $[L-U]_{J_{2n+1}}$ where L

is an $(n+1)$–component Brunnian link and U is the $(n+1)$–component unlink [14]. Further, we constructed a linear map

$$h_n\colon\ \mathcal{A}^c_{n-1}(\emptyset) \longrightarrow \mathrm{Br}(\overline{J}_{2n}(n+1)),$$

where $\mathcal{A}^c_{n-1}(\emptyset)$ is a $\mathbf{Z}$–module of connected trivalent diagrams with $2n-2$ vertices. h_n is an isomorphism over $\mathbf{Q}$ for $n \geq 2$. See Section 5 for precise definitions.

Let $\overline{\mathcal{S}}_k$ be the abelian group of Y_{k+1}–equivalence classes of integral homology spheres which are Y_k–equivalent to S^3. $\overline{\mathcal{S}}_{2k+1}=0$ for all $k\geq 1$, and it is well known that $\overline{\mathcal{S}}_{2k}$ is isomorphic to $\mathcal{A}^c_k(\emptyset)$ when tensoring by $\mathbf{Q}$. See Section 6.3. There is therefore an isomorphism over $\mathbf{Q}$ from $\mathrm{Br}(\overline{J}_{2n}(n+1))$ to $\overline{\mathcal{S}}_{2n-2}$, for $n\geq 2$. The next theorem states that this isomorphism is induced by $(+1)$–framed surgery.

For a null-homologous ordered link L in a compact connected oriented 3–manifold M, denote by $(L,+1)$ the link L with all components having framing $+1$.

Theorem 1.3 *For $n\geq 2$, the assignment*

$$[L-U]_{J_{2n+1}} \mapsto [S^3_{(L,+1)}]_{Y_{2n-1}}$$

defines an isomorphism

$$\kappa_n\colon\ \mathrm{Br}(\overline{J}_{2n}(n+1))\otimes\mathbf{Q} \longrightarrow \overline{\mathcal{S}}_{2n-2}\otimes\mathbf{Q}.$$

We actually show that these two $\mathbf{Q}$–modules are isomorphic to the so-called ‘connected part’ of the Ohtsuki filtration, by using the abelian group $\mathcal{A}^c_{n-1}(\emptyset)$. See Section 6 for definitions and statements.

The rest of this paper is organized as follows.

In Section 2, we give a brief review of the theory of claspers, both for compact connected oriented 3–manifolds and for links in a fixed manifold. In Section 3, we study the Y_k–equivalence class of integral homology spheres obtained by surgery along claspers with several *special leaves*. This section can be read separately from the rest of the paper and might be of independent interest. In Section 4, we use the main result of section 3 to prove Theorems 1.1 and 1.2. In Section 5, we recall several results obtained by Habiro and the author in [14]. In Section 6, we define the material announced above and prove Theorem 1.3. In Section 7, we give the (technical) proof of Proposition 3.8.

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2 Claspers

Throughout this paper, all 3–manifolds will be supposed to be compact, connected and oriented.

2.1 Clasper theory for 3–manifolds

Let us briefly recall from [4, 7, 12] the fundamental notions of clasper theory for 3–manifolds.

Definition 2.1 A *clasper* in a 3–manifold M is an embedding

$$G\colon\ F \longrightarrow \operatorname{int} M$$

of a compact (possibly unorientable) surface F. F is decomposed into *constituents* connected by disjoint bands called *edges*. Constituents are disjoint connected subsurfaces, either annuli or disks:

- A *leaf* is an annulus with one edge attached.
- A *node* is a disk with three edges attached.
- A *box* is a disk with *at least* three edges attached, one being distinguished with the others. This distinction is done by drawing a box as a rectangle.

Observe that this definition slightly extends the one in [12], where a box has always three edges attached.

We will make use of the drawing convention for claspers of [12, Figure 7], except for the following: a $\oplus$ (resp. $\ominus$) on an edge represents a positive (resp. negative) half-twist. This replaces the convention of a circled S (resp. S^{-1}) used in [12].

2.1.1 Surgery along claspers Given a clasper G in M, we can construct, in a regular neighborhood of the clasper, an associated framed link L_G as follows. First, replace each node and box of G by leaves as shown in Figure 2.1 (a) and (b). We obtain a union of I–shaped claspers, one for each edge of G. L_G is obtained by replacing each of these I–shaped claspers by a 2–component framed link as shown in Figure 2.1 (c).[1]

Surgery along the clasper G is defined to be surgery along L_G.

In [12, Proposition 2.7], Habiro gives a list of 12 moves on claspers which gives *equivalent* claspers, that is claspers with diffeomorphic surgery effect. We will freely

[1] Here and throughout the paper, blackboard framing convention is used.

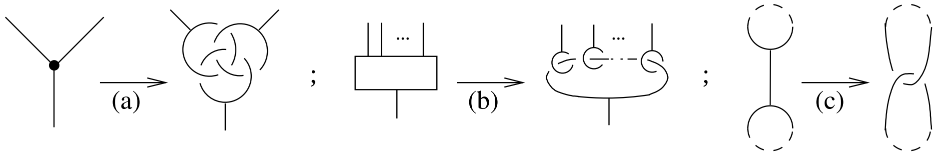

Figure 2.1: Constructing the framed link associated to a clasper

use *Habiro's moves* (which are essentially derived from Kirby calculus) by referring to their numbering in Habiro's paper.

2.1.2 The Y_k–equivalence For $n \geq 1$, a *Y_n–graph* is a connected clasper G without boxes and with n nodes, where a connected clasper is a clasper whose underlying surface is connected. The integer n is called the *degree* of G.

A *Y_k–tree* is a Y_k–graph T such that the union of edges and nodes of T is simply connected. For $k \geq 3$, we say that a Y_k–tree T in a 3–manifold M is *linear* if there is a 3–ball in M which intersects the edges and nodes of T as shown in Figure 2.2. The leaves denoted by f and f' in the figure are called the *ends* of T.

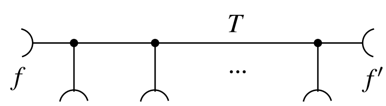

Figure 2.2: A linear tree T and its two ends f and f'

A *Y_k–forest* is a clasper $T = T_1 \cup \ldots \cup T_p$ $(p \geq 0)$, where T_i is a Y_{k_i}–tree $(p \geq i \geq 1)$, such that $min_{1 \leq i \leq p} k_i = k$.

A *Y_k–subtree* T of a clasper G is a connected union of leaves, nodes and edges of G such that the union of edges and nodes of T is simply connected and such that T intersects $\overline{G \setminus T}$ along the attaching region of some edges of T, called *branches*.

A surgery move on M along a Y_k–graph G is called a *Y_k–move*. For example, a Y_1–move is equivalent to Matveev's Borromean surgery [16].

The *Y_k–equivalence* is the equivalence relation on 3–manifolds generated by Y_k–moves and orientation-preserving diffeomorphisms. This equivalence relation becomes finer as k increases: if $k \leq l$ and if $M \sim_{Y_l} N$, then we also have $M \sim_{Y_k} N$.

Recall that 'trees do suffice to define the Y_k–equivalence'. That is, $M \sim_{Y_k} N$ implies that there exists a Y_k–forest F in M such that $M_F \cong N$.

2.2 Clasper theory for links

Another aspect of the theory of claspers is that it allows to study links in a *fixed* manifold. For this we use a slightly different type of claspers.

Definition 2.2 Let L be a link in a 3–manifold M, and let G be a clasper in M which is disjoint from L. A *disk-leaf* of G is a leaf l of G which is an unknot bounding a disk D in M with respect to which it is 0–framed.[2] We call D the *bounding disk* of f. The interior of D is disjoint from G and from any other bounding disk, but it may intersect L transversely. For convenience, we say that a disk-leaf f *intersects* the link L when its bounding disk does.

A C_n*–tree* (resp. *linear* C_n*–tree*) for a link L in a 3–manifold M is a Y_{n-1}–tree (resp. *linear* Y_{n-1}*–tree*) in M such that each of its leaves is a disk-leaf.

Given a C_n–tree C in M, there exists a canonical diffeomorphism between M and the manifold M_C. So surgery along a C_n–tree can be regarded as a local move on links in the manifold M.

A C_n–tree G for a link L is *simple* (with respect to L) if each disk-leaf of G intersects L exactly once.

A surgery move on a link L along a C_k–tree is called a C_k*–move*. The C_k*–equivalence* is the equivalence relation on links generated by the C_k–moves and isotopies. As in the case of manifolds, the C_n–equivalence relation implies the C_k–equivalence if $1 \leq k \leq n$. For more details, see [8, 12].

2.3 Some technical lemmas

In this subsection, we state several technical lemmas about claspers.

First, we introduce several moves on claspers which produce equivalent claspers, like the 12 Habiro's moves. In each of the next three statements, the figure represents two claspers in a given 3–manifold which are identical outside a 3–ball, where they are as depicted.

Lemma 2.3 *The move of Figure 2.3 produces equivalent claspers.*

[2] Here we regard a leaf, which is an embedded annulus, as a knot with a framing.

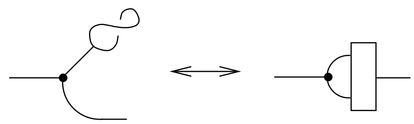

Figure 2.3

This is an immediate consequence of [4, Theorem 3.1] (taking into account that the convention used in [4] for the definition of the surgery link associated to a clasper is the opposite of the one used in the present paper).

Lemma 2.4 *The move of Figure 2.4 produces equivalent claspers.*

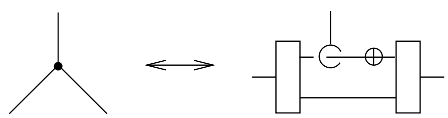

Figure 2.4

This move is, in some sense, the inverse of Habiro's move 12. See also Figure 25 of [3], where a similar move appears.

Proof Consider the clasper on the right-hand side of Figure 2.4. By replacing the two boxes by leaves as shown in Figure 2.1 (b) and applying Habiro's move 1, we obtain the clasper depicted on the left-hand side of Figure 2.5. Now, the three leaves depicted

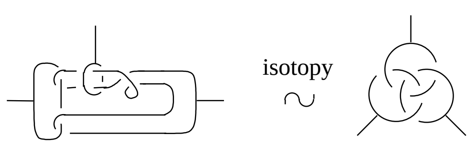

Figure 2.5

in this figure form a 3–component link which is isotopic to the Borromean link. As shown in Figure 2.1 (a), this is equivalent to a node. □

Lemma 2.5 *The moves of Figure 2.6 produce equivalent claspers.*

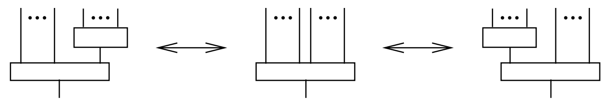

Figure 2.6: The associativity of boxes

This 'associativity' property of boxes is easily checked using Figure 2.1 (b) – see Figure 37 of [12].

The next lemma deals with crossing change operations on claspers. A crossing change is a local move as illustrated in Figure 2.7. The proof is omitted, as it uses the same techniques as in [12, Section 4] (where similar statements appear). See also [17, Section 1.4].

Lemma 2.6 *Let $T_1 \cup T_2$ be a disjoint union of a Y_{k_1}–tree and a Y_{k_2}–tree in a 3–manifold M. Let $T_1' \cup T_2'$ be obtained by a crossing change c of an edge* or *a leaf of T_1 with an*

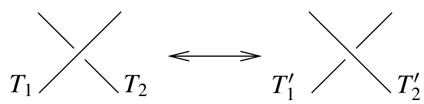

Figure 2.7: A crossing change

edge or *a leaf of T_2 (see Figure 2.7), and let $C \in \{0,1,2\}$ denotes the number of edges involved in the crossing change c. Then*

(1) $M_{T_1 \cup T_2} \sim_{Y_{k_1+k_2+C}} M_{T_1' \cup T_2'}$.

(2) $M_{T_1 \cup T_2} \sim_{Y_{k_1+k_2+C+1}} M_{T_1' \cup T_2' \cup T}$*, where T is a parallel copy, disjoint from $T_1' \cup T_2'$, of some $Y_{k_1+k_2+C}$–tree $\tilde{T}$ obtained as follows:*

(a) *If c involves an edge e_1 of T_1 and an edge e_2 of T_2, then $C = 2$ and $\tilde{T}$ is obtained by inserting a node n_1 in e_1 and a node n_2 in e_2, and connecting n_1 and n_2 by an edge.*

(b) *If c involves an edge e of T_1 and a leaf f of T_2, then $C = 1$ and $\tilde{T}$ is obtained by inserting a node n in e, and connecting n_1 to the edge incident to f.*

(c) *If c involves a leaf f_1 of T_1 and a leaf f_2 of T_2, then $C = 0$ and $\tilde{T}$ is obtained by connecting the edges incident to f_1 and f_2.*

Remark 2.7 This lemma is only valid for trees. However, if we are given graphs or subtrees instead, observe that it suffices to use Habiro's move 2 to obtain equivalent

trees. So in this paper, whenever we apply Lemma 2.6 to graphs or subtrees, it implicitly means that we apply the lemma to some equivalent trees obtained by Habiro's move 2.

The next result follows from Lemma 2.6 and [12, Proposition 2.7]. See also [4, 20].

Lemma 2.8 *Let G be a Y_k–tree in a 3–manifold M, and let G_+ be a Y_k–tree obtained from G by inserting a positive half twist in an edge. Then*

$$M_{G\cup\tilde{G}_+} \sim_{Y_{k+1}} M,$$

where $\tilde{G}_+$ is obtained from G_+ by an isotopy so that it is disjoint from G.

2.4 The IHX relation for Y_k–graphs

We have the following version of the IHX relation for Y_k–graphs.

Lemma 2.9 *Let I, H and X be three Y_k–graphs in a 3–manifold M, which are identical except in a 3–ball where they look as depicted in Figure 2.8. Then*

$$M_I \sim_{Y_{k+1}} M_{H\cup\tilde{X}},$$

where $\tilde{X}$ is obtained from X by an isotopy so that it is disjoint from H.

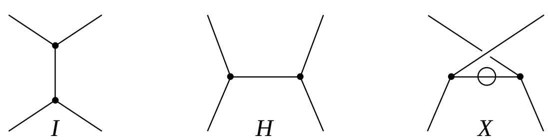

Figure 2.8: The three Y_k–graphs I, H and X

Various similar statements appear in the literature. For example, an IHX relation is proved in [4] at the level of finite type invariants, in [3] for C_n–trees (see also [8]), and in [20, pages 397–398] for Y_n–graphs without leaves.

Proof For simplicity, we give the proof for the case of Y_2–trees. In the general case, the proof uses the same arguments as below, together with the zip construction ([12, Section 3], see also [3, Section 4.2]).

Consider the Y_2–tree I, and apply Lemma 2.4 at one of its nodes. Then, apply Habiro's move 11 so that we obtain the clasper $G_1 \sim I$ depicted in Figure 2.9. By an isotopy and Habiro's move 7, G_1 is seen to be equivalent to the clasper G_2 of Figure 2.9. Consider

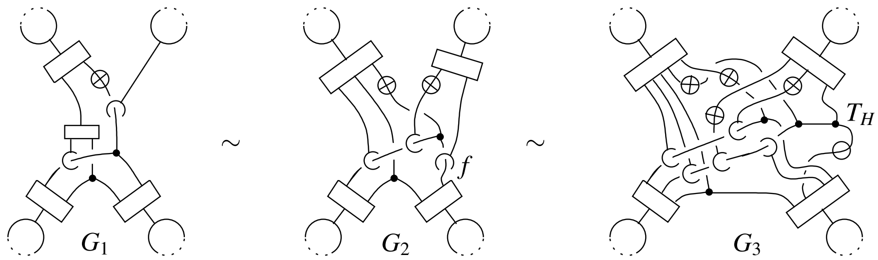

Figure 2.9

the leaf of G_2 denoted by f in the figure. By an application of Habiro's move 12 at f, followed by moves 7 and 11, we obtain the clasper G_3 of Figure 2.9. Observe that G_3 contains a Y_2–subtree T_H. By Habiro's move 6, Lemma 2.8 and Lemma 2.6 (1), we have

$$M_{G_3} \sim_{Y_3} M_{H\cup G_4},$$

where G_4 is the clasper depicted in Figure 2.10. Now, consider the leaf f' of G_4 (see

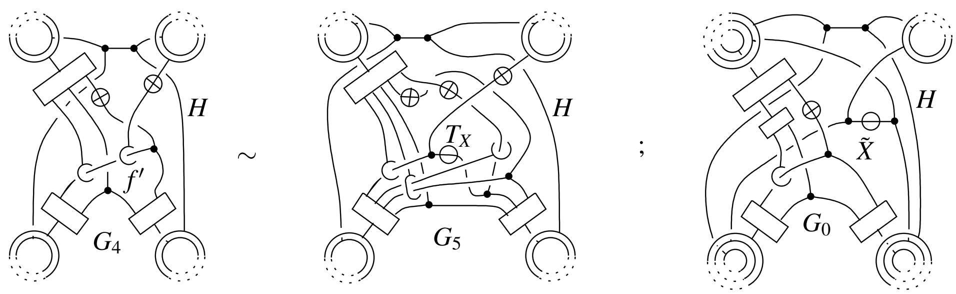

Figure 2.10

the figure). Apply Habiro's move 12 at f' and moves 7 and 11, just as we did previously for the clasper G_2. The resulting clasper $G_5 \sim G_4$ contains a Y_2–subtree T_X (see Figure 2.10). As above, we obtain by Lemmas 2.8 and 2.6 (1):

$$M_{H\cup G_5} \sim_{Y_3} M_{H\cup\tilde{X}\cup G_0},$$

where G_0 is represented in the right-hand side of Figure 2.10. By Habiro's moves 11 and 4, we obtain that $M_{H\cup\tilde{X}\cup G_0} \cong M_{H\cup\tilde{X}}$. □

One can check the following slightly stronger version of Lemma 2.9 when I, H and X are three Y_k–trees (Note that Habiro's move 2 always allows us to have this condition satisfied).

Lemma 2.10 *Let I, H, X and $\tilde{X}$ be four Y_k–trees in a 3–manifold M as in Lemma 2.9. Then*

$$M_I \sim_{Y_{k+2}} M_{H \cup \tilde{X} \cup F},$$

where F is a union of disjoint Y_{k+1}–trees. Each Y_{k+1}–tree T in F is obtained from either H or X by taking a parallel copy f of one of its leaves, inserting a node n in one of its edges, connecting n and f by an edge, and performing an isotopy so that T is disjoint from H, $\tilde{X}$ and $F \setminus T$.

Consider for example the case of Y_2–trees, as in the proof of Lemma 2.9. We saw there that $M_I \cong M_{G_3} \sim_{Y_3} M_{H \cup G_4}$, where G_3 and G_4 are depicted in Figure 2.9 and 2.10. Observe that $H \cup G_4$ is obtained from G_3 by several Habiro's moves and three crossing changes between an edge of the Y_2–subtree T_H and some leaf of G_3. So by (2) of Lemma 2.6 (and Habiro's move 5) one can check that

$$M_{G_3} \sim_{Y_4} M_{H \cup G_4 \cup F'},$$

where F' consists of three Y_3–trees obtained as described in the statement of the Lemma. For similar reasons, (2) of Lemma 2.6 implies that the clasper $G_5 \sim G_4$ depicted in Figure 2.10 satisfies $M_{H \cup G_5} \sim_{Y_4} M_{H \cup \tilde{X} \cup G_0 \cup F''}$, where F'' is a union of Y_3–trees of the desired form. This implies Lemma 2.10 for $k = 2$.

3 Surgery along Y_n–trees with special leaves

In this section, we study 3–manifolds obtained by surgery along Y_n–trees containing a particular type of leaves.

3.1 m–special leaves

Suppose we are given a clasper G in a 3–manifold M.

Definition 3.1 Let $m \in \mathbf{Z}$. An *m–special leaf with respect to G* is a leaf f of G which is an unknot bounding a disk D in M with respect to which it is m–framed,[3] such that the interior of D is disjoint from $G \setminus f$. D is called the *bounding disk* of f. Two bounding disks are required to be disjoint. A regular neighborhood of the union of G and the bounding disks is called an *s–regular neighborhood* of G.

[3] Here, as in Definition 2.2, we regard a leaf as a knot with a framing.

In particular, a 0–special leaf with respect to G is called a *trivial leaf*. If a Y_k–graph G in M contains a 0–special leaf f with respect to G, then M_G is diffeomorphic to M [12, 4].

In the rest of the paper, a *special leaf* is an m–special leaf for some unspecified integer m.[4] The mention 'with respect to' will be omitted when G is clear from the context.

3.2 Statement of the result

Let G be a Y_n–tree in a 3–manifold M, $n \geq 2$. It is well-known that, if G contains a (-1)–special leaf, then

$$M_G \sim_{Y_{n+1}} M. \tag{3–1}$$

See [20, Lemma E.21] for a proof for $M = S^3$, which can be generalized to our context. See also [4, Lemma 4.9].

We obtain the following generalization.

Theorem 3.2 *Let G be a Y_n–tree in a 3–manifold M, with $n \geq 2$. Let l denote the number of special leaves with respect to G. Then*

(1) *If $l < n$, then $M_G \sim_{Y_{n+l}} M$.*

(2) *If $l = n$, then $M_G \sim_{Y_{2n-1}} M$.*

(3) *If $l > n$, then $M_G \sim_{Y_{2n}} M$.*

The proof is given in Section 3.6. In the next three subsections, we prove Theorem 3.2 in several important cases and provide a lemma which is used in Section 3.6.

3.3 The case of a tree with one special leaf

In this subsection, we prove Theorem 3.2 for $l = 1$.

Lemma 3.3 *Let G be a Y_n–tree in a 3–manifold M, with $n \geq 2$. Suppose that G contains an m–special leaf ; $m \in \mathbf{Z}$. Then $M_G \sim_{Y_{n+1}} M$.*

[4] Note that in some literature [4] the terminology 'special leaf' is used to denote a (-1)–special leaf.

Proof We first prove the lemma for all $m < 0$, by induction. As recalled in Section 3.2, we already have the result for $m = -1$. Now consider a Y_n–tree G in M with an m–special leaf f, $m < 0$. Denote by G' the clasper obtained by replacing f by the union of a box b and two edges e_1 and e_2 connecting b respectively to a (-1)–special leaf f_1 and a $(m+1)$–special leaf f_2 (both leaves being special with respect to G'). By Habiro's move 7, $G' \sim G$. Denote by G_i the Y_n–tree in M obtained from G by replacing f by f_i $(i = 1, 2)$. By a zip construction, we have

$$G' \sim (G_1 \cup P),$$

where P satisfies $P \sim G_2$. By (3–1) it follows that $M_G \sim_{Y_{n+1}} M_{G_2}$. The result then follows from the induction hypothesis.

Similarly, it would suffice to show the result for $m = 1$ to obtain, by a similar induction, the result for all $m > 0$. For this, consider the case $m = 0$. In this case, f is a trivial leaf and therefore $M_G \cong M$. The same construction as above, with a (-1)–special leaf f_1 and a 1–special leaf f_2, shows that $M \sim_{Y_{n+1}} M_{G'}$, where G' is a Y_n–tree in M with a 1–special leaf. This concludes the proof. □

3.4 The case of a Y_2–tree

In this section, we prove Theorem 3.2 for $n = 2$. The proof mainly relies on the following lemma.

Lemma 3.4 *Let G be a Y_2–tree in a 3–manifold M which contains two (-1)–special leaves which are connected to the same node. Then $M_G \sim_{Y_4} M$.*

Proof Denote by w the node of G which is connected to the two special leaves. w is connected by an edge to another node v. By applying Lemma 2.4 at v, G is equivalent, in an s–regular neighborhood, to a clasper G' which is identical to G, except in a 3–ball where it is as depicted in Figure 3.1 (a). There, the node w' corresponds to the node w of G. By Lemma 2.3 and Habiro's move 6, we obtain the clasper depicted in Figure 3.1 (b), which is equivalent to the one depicted in Figure 3.1 (c) by three applications of Habiro's move 12, Lemma 2.5 and an isotopy. Denote by G'' this latter clasper. As the figure shows, G'' contains a Y_4–subtree T. Actually, T is a 'good input subtree' of G'', in the sense of [12, Definition 3.13]. Denote by $\tilde{G}''$ the clasper obtained from G'' by inserting in each branch of T a pair of small Hopf-linked leaves. By Habiro's move 2, $\tilde{G}'' \sim G'$. Denote by $\tilde{T}$ the Y_4–tree of $\tilde{G}''$ which corresponds to T. By an application of the zip construction, we obtain $M_{G''} \sim_{Y_4} M_{\tilde{G}'' \setminus \tilde{T}}$. Further, it follows from Habiro's moves 3 and 4 that $\tilde{G}'' \setminus \tilde{T} \sim \emptyset$. □

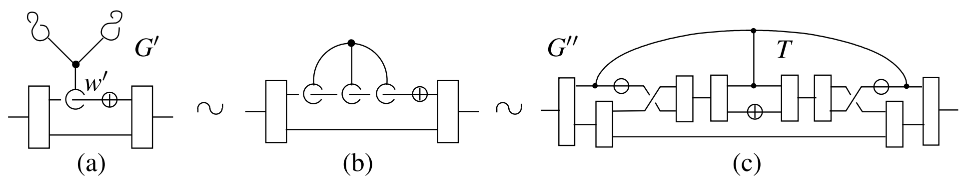

Figure 3.1

The following technical lemma will allow us to generalize Lemma 3.4 to arbitrary special leaves.

Lemma 3.5 *Let G be a Y_2–tree in a 3–manifold M which contains two special leaves which are connected to the same node. Then*

$$M_G \sim_{Y_4} M_{G_1 \cup \tilde{G}_2},$$

where, for $i = 1, 2$, G_i is obtained by replacing a k–special leaf of G by a k_i–special leaf, such that $k_1 + k_2 = k$, and where $\tilde{G}_2$ is obtained from G_2 by an isotopy so that it is disjoint from G_1.

Proof Denote respectively by f and f' the k–special (resp. k'–special) leaf of G, $k, k' \in \mathbf{Z}$. Just as in the proof of Lemma 3.3, we can use Habiro's moves 7 and the zip construction to see that G is equivalent, in an s–regular neighborhood, to the clasper C_1 of Figure 3.2, where f_1 is a k_1–special leaf and f_2 is a k_2–special leaf such that $k_1 + k_2 = k$. Consider the leaf of C_1 denoted by F in the figure. By Habiro's move 12 at F, followed by two applications of Habiro's move 11, we have $C_1 \sim C_2$, where C_2 is represented in Figure 3.2.

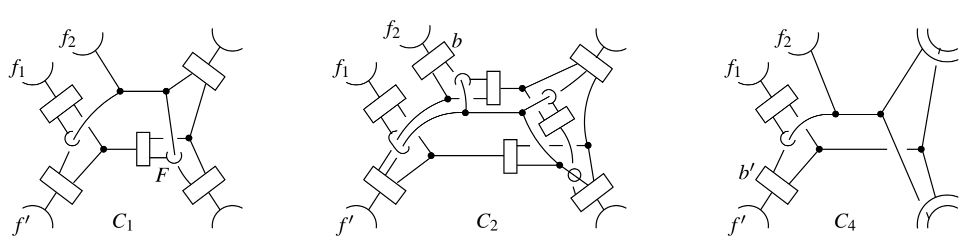

Figure 3.2

Consider the box b of C_2 (see Figure 3.2). By applying Habiro's move 5 at b, C_2 is equivalent to a clasper containing a Y_3–subtree T and a Y_1–subtree T' such that both T

and T' contain a copy of f_2. Denote by C_3 the clasper obtained by replacing these two (linked) copies of f_2 by two k_2–special leaves. By Lemma 2.6, we have $M_{C_2} \sim_{Y_4} M_{C_3}$. It follows from Lemma 3.3 and Habiro's move 5 that $M_{C_3} \sim_{Y_4} M_{C_4}$, where C_4 is as represented in Figure 3.2. By applying Habiro's move 5 at the box b', C_4 is equivalent to a clasper containing a Y_2–tree and a Y_2–subtree, each containing a copy of f'. By Lemma 2.6, $M_{C_4} \sim_{Y_4} M_{C_5}$, where C_5 is obtained by replacing these two (linked) copies of f' in C_4 by two k'–special leaves. The result then follows from an isotopy and Habiro's move 3. □

We can now prove the case $n = 2$ of Theorem 3.2.

Let G be a Y_2–tree in a 3–manifold M with l special leaves. If $l = 0$, then the result is obvious. If $l = 1$, Lemma 3.3 implies that $M_G \sim_{Y_3} M$. If $l = 2$, then $M_G \sim_{Y_3} M$ also follows from Lemma 3.3. It remains to prove the result when $l = 3$ or 4.

Let $k, k' \in \mathbf{Z}$. Denote by $G_{k,k'}$ a Y_2–tree in M containing a k–special leaf f and an k'–special leaf f', both connected to the same node. Observe that it suffices to show that

$$M_{G_{k,k'}} \sim_{Y_4} M \tag{3–2}$$

If $k = k' = -1$, then (3–2) follows from Lemma 3.4. Now, let us fix $k' = -1$. Then we can show by induction that (3–2) holds for all $k < -1$. Indeed, consider some integer $m < -1$, and consider $G_{m,-1}$ in M. By Lemma 3.5, we have

$$M_{G_{m,-1}} \sim_{Y_4} M_{C_1 \cup C_2},$$

where C_1 contains two (-1)–special leaves connected to the same node, and where C_2 contains a (-1)–special leaf and an $m+1$–special leaf, both connected to the same node. By Lemma 3.4 and the induction hypothesis, we thus obtain $M_{G_{m,-1}} \sim_{Y_4} M$.

So we can now set k' to be any negative integer, and prove (3–2) for all $k < -1$, by strictly the same induction.

Similarly, it would suffice to show the result for $G_{1,1}$ to be able to prove (3–2) for all $k, k' > 0$. Consider $G_{0,-1}$ in M. In this case, f is a trivial leaf and $M_{G_{0,1}} \cong M$. By applying Lemma 3.5 at f,

$$M_{G_{0,1}} \cong M \sim_{Y_4} M_{G_1 \cup G_2},$$

where G_1 (resp. G_2) contains a (-1)–special leaf and a 1–special (resp. (-1)–special) leaf, both connected to the same node. It follows from Lemma 3.3 that $M \sim_{Y_4} M_{G_1}$. This proves (3–2) for $k = 1$ and $k = -1$. We obtain (3–2) for $k = k = 1$ similarly, by applying Lemma 3.5 to $G_{0,1}$ in M.

3.5 The cutting lemma.

Let G be a Y_n–tree in M, with $n \geq 3$. By inserting a pair of small Hopf-linked leaves in an edge of G, we obtain a Y_{n_1}–tree G_1 and a Y_{n_2}–tree G_2 such that $n_1+n_2=n$ and $G_1 \cup G_2 \sim G$ (by Habiro's move 2). See Figure 3.3.

Lemma 3.6 *Let $i=1,2$. Suppose that, in a regular neighborhood N_i of G_i, we have $(N_i)_{G_i} \sim_{Y_{k_i}} N_i$, with $k_1 \geq 2$ and $k_2 \geq 1$. Then*

(1) *$M_G \sim_{Y_{k_1+2}} M$, if G_2 is a Y_1–tree containing at least one special leaf with respect to $G_1 \cup G_2$,*

(2) *$M_G \sim_{Y_{k_1+k_2}} M$, otherwise.*

Proof Denote by N an s–regular neighborhood of $G \sim G_1 \cup G_2$. Consider a 3–ball B in M which intersects N and $G_1 \cup G_2$ as depicted in Figure 3.3 (a). Denote by N'

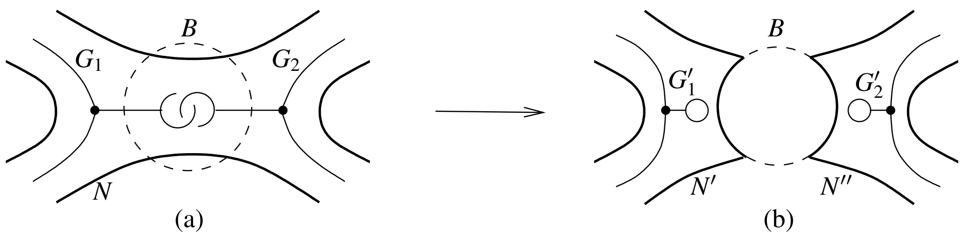

Figure 3.3

and N'' the two connected components of the closure of $N \setminus (B \cap N)$. By one crossing change and isotopy, we can homotop the two Hopf-linked leaves of $G_1 \cup G_2$ into $N \setminus (B \cap N)$ so that, if $G'_1 \cup G'_2$ denotes the resulting clasper, we have $G'_1 \subset N'$ and $G'_2 \subset N''$. See Figure 3.3 (b). Each of G'_1 and G'_2 contains a trivial leaf with respect to $G'_1 \cup G'_2$, so we have $G'_1 \cup G'_2 \sim \emptyset$ in N.

We now prove (1): suppose that G_2 contains one node and at least one special leaf with respect to $G_1 \cup G_2$. Denote by f the leaf of G_2 which forms a Hopf link with a leaf of G_1. By assumption, G_1 can be replaced by a Y_{k_1}–forest F_1 in an s–regular neighborhood N_1 so that $F_1 \cup G_2 \sim G$ in N. Consider a disk d bounded by f such that d intersects transversally edges and leaves of components of F_1. By a sequence of crossing changes, we can homotop these edges and leaves into $N' \subset N$: the clasper G' obtained from $F_1 \cup G_2$ by this homotopy satisfies $G' \sim G'_1 \cup G'_2 \sim \emptyset$ in N. So it would suffice to show that $M_{F_1 \cup G_2} \sim_{Y_{k_1+2}} M_{G'}$.

By Lemma 2.6, we have $M_{F_1 \cup G_2} \sim_{Y_{k_1+2}} M_{\tilde{F}_1 \cup \tilde{G}_2}$, where $\tilde{F}_1 \cup \tilde{G}_2$ is obtained by 'homotoping' into N' all edges of F_1 and all Y_k–trees of F_1 with $k > k_1$. Denote by $\tilde{f}$ the leaf of $\tilde{G}_2$ corresponding to f. There is a sequence of crossing changes

$$\tilde{F}_1 \cup \tilde{G}_2 = C_0 \mapsto C_1 \mapsto C_2 \mapsto \ldots \mapsto C_{p-1} \mapsto C_p = G',$$

where, for each $1 \le k \le p$, C_k is obtained from C_{k-1} by one crossing change between $\tilde{f}$ and a leaf l of a Y_{k_1}–tree T_k of $\tilde{F}_1$.[5] By Lemma 2.6, we have $M_{C_k} \sim_{Y_{k_1+2}} M_{C_{k-1} \cup H_k}$, where H_k is a Y_{k_1+1}–tree obtained by connecting the edges of $\tilde{G}_2$ and T_k attached to $\tilde{f}$ and l respectively. In particular, H_k contains a special leaf with respect to $C_{k-1} \cup H_k$. So by Lemma 3.3, we have $M_{C_k} \sim_{Y_{k_1+2}} M_{C_{k-1}}$. It follows that $M_{\tilde{F}_1 \cup \tilde{G}_2} \sim_{Y_{k_1+2}} M_{G'}$, which concludes the proof of (1).

The proof of (2) is simpler, and left to the reader. It uses exactly the same arguments as above, by considering the Y_{k_i}–forest F_i $(i = 1, 2)$ in an s–regular neighborhood N_i of G_i such that $F_1 \cup F_2 \sim G$ in N. □

3.6 Proof of Theorem 3.2

Suppose that G is a Y_n–tree in M with l special leaves ; $n \ge 2$, $l \ge 0$.

3.6.1 The case $l < n$ In this case, it is necessary to reduce the problem to linear trees. We have the following.

Claim 3.7 Let $1 \le p \le l$ be an integer. Pick two non-special leaves f_1 and f_2 of G. Then we have, by successive applications of the IHX relation,

$$M_G \sim_{Y_{n+p}} M_{L_p},$$

where L_p is a union of disjoint linear Y_k–trees with $n \le k \le n + p - 1$ such that

- the ends of each linear tree are parallel copies of f_1 and f_2,
- each Y_k–tree contains $(n + l - k)$ special leaves with respect to L_p.

Proof of the claim The claim is proved by induction on p. Observe that we can use the IHX relation to replace T by a union L_1 of linear Y_k–trees whose ends are parallel copies of f_1 and f_2. Lemma 2.6 (1) ensures that each tree has l special leaves with respect to L_1. This proves the case $p = 1$. Now assume the claim for some $p \ge 1$:

[5] Here, abusing notations, we still denote by $\tilde{f}$, $\tilde{G}_2$ and $\tilde{F}_1$ the corresponding elements in C_k, for all $k \ge 1$.

$M_T \sim_{Y_{n+p}} M_{L_p}$, where L_p is as described above. By assumption, this equivalence comes from Lemma 2.9, so we can apply Lemma 2.10. There exists a union F of disjoint (possibly non linear) Y_{n+p}–trees such that $M_T \sim_{Y_{n+p+1}} M_{L_p \cup F}$. For each tree T in F, its $(n+p+2)$ leaves are obtained by taking the leaves of a Y_{n+p-1}–tree in L_p and adding a parallel copy of one of them. If this additional leaf is a copy of a special leaf f (with respect to L_p), the two (linked) copies of f in T are not special leaves with respect to $L_p \cup F$. This shows that each tree in F contains at least $(l-p)$ special leaves with respect to $L_p \cup F$. Note that each such tree also contains (at least) a copy of f_1 and f_2. So by Lemma 2.9 we have $M_{L_p \cup F} \sim_{Y_{n+p+1}} M_{L_{p+1}}$, where L_{p+1} is of the desired form. □

It follows from Claim 3.7 that

$$M_T \sim_{Y_{n+l}} M_L,$$

where L is a union of linear Y_k–trees with $n \leq k \leq n+l-1$, each such linear Y_k–tree containing (at least) $(n+l-k)$ special leaves with respect to L, and whose ends are non-special leaves.

So it suffices to prove the case $l < n$ of Theorem 3.2 for linear Y_n–trees whose ends are non-special leaves. We proceed by induction on n.

For $n = 2$, the statement follows from Section 3.4.

Now, assume that the statement holds true for all $k < n$, and consider a linear Y_n–tree G whose ends are two non-special leaves. Insert a pair of small Hopf-linked leaves in an edge of G such that it produces a union of two linear trees $G_1 \cup G_2 \sim G$ with $degG_1 = n_1$ and $degG_2 = n_2$. Denote respectively by l_1 and l_2 the number of special leaves with respect to $G_1 \cup G_2$ in G_1 and G_2. We have $n_1 + n_2 = n$ and $l_1 + l_2 = l$. Denote also by N_1 an s–regular neighborhood of G_1.

- If we can choose $n_2 = 1$ and $l_2 = 1$, then $n_1 = n-1$ and $l_1 = l-1$. So $l_1 < n_1$ and by the induction hypothesis we have $(N_1)_{G_1} \sim_{Y_{n+l-2}} N_1$ (G_1 is indeed linear). As G_2 contains one special leaf with respect to $G_1 \cup G_2$, we obtain the result by Lemma 3.6 (1).
- Otherwise, then $l < n-1$, and we can choose G_2 such that $n_2 = 1$ and $l_2 = 0$ (that is, G_2 contains one node connected to 2 non-special leaves). As $l_1 = l < n_1 = n-1$, we have $(N_1)_{G_1} \sim_{Y_{n+l-1}} N_1$ (by the induction hypothesis), and the result follows from Lemma 3.6 (2).

This completes the proof of the case $l < n$.

3.6.2 The case $l \geq n$ The case $l = n$ follows immediately from the case $l = n-1$, by regarding one of the special leaves as a leaf.

We prove the case $l = n+1$ by induction on the degree n. The case $n = 2$ was proved in Section 3.4. Consider a Y_n–tree G with $l \geq n$ special leaves. As in Section 3.6, insert a pair of Hopf-linked leaves in an edge of G so that we obtain a union of two trees $G_1 \cup G_2 \sim G$ with $degG_1 = n-1$ and $degG_2 = 1$. Denote respectively by l_1 and l_2 the number of special leaves with respect to $G_1 \cup G_2$ in G_1 and G_2. There are two cases, depending on whether $l_2 = 1$ or 2.

- If $l_2 = 1$, then $l_1 = n = n_1 + 1$, and thus, by the induction hypothesis we have $(N_1)_{G_1} \sim_{Y_{2n-3}} N_1$ in an s–regular neighborhood N_1 of G_1. The result follows from Lemma 3.6 (1).
- If $l_2 = 2$, then $l_1 = n-1 = n_1$. It thus follows from the case $l = n$ of Theorem 3.2 that $(N_1)_{G_1} \sim_{Y_{2n-3}} N_1$ in an s–regular neighborhood N_1 of G_1. The result then follows as above from Lemma 3.6 (1).

The case $l = n+2$ follows from the case $l = n+1$ by regarding one of the special leaves as a leaf.

3.7 Some special cases for Theorem 3.2

We have the following improvement of Theorem 3.2 for linear trees having only (-1)–special leaves.

Proposition 3.8 *Let G be a linear Y_n–tree in a 3–manifold M, $n \geq 2$, such that all its leaves are (-1)–special leaves. Then in an s–regular neighborhood N of G (which is a 3–ball in M) we have*

$$N_G \sim_{Y_{2n+1}} N_{\Theta_n},$$

where Θ_n is the connected Y_{2n}–graph without leaves depicted in Figure 3.4

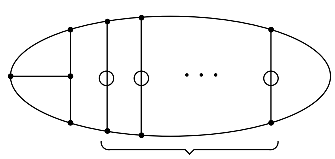

Figure 3.4: The Y_{2n}–graph Θ_n

Remark 3.9 Note that "$\sim_{Y_{2n+1}}$" in Proposition 3.8 can be replaced by "$\sim_{Y_{2n+2}}$". This follows from the fact that if two integral homology balls are Y_{2n+1}–equivalent then they are Y_{2n+2}–equivalent ($n > 1$).

The proof of Proposition 3.8 uses rather involved calculus of claspers, and is therefore postponed to Section 7. Note that this result is not needed for the rest the paper. A reader who is not too comfortable with claspers (but who nevertheless reached this point) may thus safely skip this proof.

Also, one can check that if G is a Y_n–tree in a 3–manifold M with n special leaves, we have

$$M_G \sim_{Y_{2n}} M \tag{3–3}$$

in the two following situations:

- G contains a $2k$–special leaf, for some integer k.
- The homology class in $H_1(M; \mathbf{Z}/2\mathbf{Z})$ of a non-special leaf of G is zero.[6] In particular, (3–3) always holds if $M = S^3$.

4 Y_k–equivalence for 3–manifolds obtained by surgery along Brunnian links

In this section, we prove Theorems 1.1 and 1.2. The proofs use a characterization of Brunnian links in terms of claspers due to Habiro, and independently to Miyazawa and Yasuhara, which involves the notion of C_k^a–equivalence. Let us first recall from [11] the definition and some properties of this equivalence relation.

4.1 C_k^a–equivalence

Definition 4.1 Let L be an m–component link in a 3–manifold M. For $k \geq m-1$, a *C_k^a–tree* for L in M is a C_k–tree T for L in M, such that

(1) all the strands intersecting a given disk-leaf of T are from the same component of L,

(2) T intersects *all* the components of L.

[6] This fact was pointed out to the author by Kazuo Habiro.

A (simple) C_k^a*–forest* L is a clasper consisting only of (simple) C_k^a–tree for L.

A C_k^a*–move* on a link is surgery along a C_k^a–tree. The C_k^a*–equivalence* is the equivalence relation on links generated by C_k^a–moves.

The main tool in the proofs of Theorems 1.1 and 1.2 is the following.

Theorem 4.2 [11, 18] *Let L be an $(n+1)$–component link in S^3. L is Brunnian if and only if it is C_n^a–*equivalent *to the $(n+1)$–component unlink U.*

In the proof of Theorem 1.2, we will also need the next result.

Theorem 4.3 ([18], see also [13]) *Two $(n+1)$–component Brunnian links in S^3 are link-homotopic if and only if they are C_{n+1}^a–equivalent.*

Note that this statement does not appear explicitly in [18]. However, it is implied by the proof of [18, Theorem 3]. An alternative proof was given subsequently by Habiro and the author [13].

4.2 Proof of Theorem 1.1

Let $m=(m_1,...,m_{n+1})\in\mathbf{Z}^{n+1}$, $n\geq 2$ and let L be an $(n+1)$–component Brunnian link in a 3–manifold M. By Theorem 4.2, L is C_n^a–equivalent to an $(n+1)$–component unlink U in M. So by [11, Lemma 7] there exists a simple C_n^a–forest $F=T_1\cup...\cup T_p$ for U such that $L\cong U_F$. We thus have

$$M_{(L,m)}\cong M_{G_m(F)},$$

where $G_m(F)$ is the clasper obtained from F by performing $\frac{1}{m_i}$–framed surgery along the i^{th} component U_i of U for all $1\leq i\leq n+1$. Indeed, $\frac{1}{m_i}$–surgery along an unknot does not change the diffeomorphism type of M, and can be regarded as a move on claspers in M. Observe that $\frac{1}{m_i}$–surgery along U_i turns each disk-leaf of F intersecting U_i into a $(-m_i)$–framed unknot (here, we forget the bounding disk). Thus $\frac{1}{m}$–surgery along U turns each C_n^a–tree T_j of F into a Y_{n-1}–tree G_j in M. However, the $(n+1)$ corresponding leaves of G_j might not be special leaves with respect to $G_m(F)$, as they can be linked with the leaves of other components of $G_m(F)$. Lemma 2.6 (1) can be used to unlink these leaves 'up to Y_{2n-2}–equivalence'. Namely, Lemma 2.6 implies that $M_{G_m(F)}\sim_{Y_{2n-2}}M_{\tilde{G}_m(F)}$, where $\tilde{G}_m(F)$ is a union of Y_{n-1}–trees, each containing $(n+1)$ special leaves with respect to $\tilde{G}_m(F)$. The result then follows from Theorem 3.2.

4.3 Proof of Theorem 1.2

Let L and L' be two link-homotopic $(n+1)$–component Brunnian links in M, and let U denote an $(n+1)$–component unlink U in M. By Theorems 4.2 and 4.3, and [11, Lemma 7], there exists a simple C^a_{n+1}–forest $F = T_1 \cup \ldots \cup T_p$ and a simple C^a_n–forest $F' = T'_1 \cup \ldots \cup T'_q$ for U such that $L' \cong U_{F'}$ and $L \cong U_{F \cup F'}$.

For all j, denote by G'_j (resp. G_j) the Y_{n-1}–tree (resp. Y_n–tree) obtained from T'_j (resp. T_j) by $\frac{1}{m}$–surgery along U. By Lemma 2.6,

$$M_{(L,+1)} \sim_{Y_{2n-1}} M_{G'_1 \cup \ldots \cup G'_q} \sharp S^3_{G_1} \sharp \ldots \sharp S^3_{G_p} \cong M_{(L',+1)} \sharp S^3_{G_1} \sharp \ldots \sharp S^3_{G_p}.$$

So proving that $S^3_{G_i} \sim_{Y_{2n-1}} S^3$ for all $1 \leq i \leq p$ would imply the theorem.

By strictly the same arguments as in Section 4.2, the Y_n–tree G_i contains at least n special leaves, for all $1 \leq i \leq p$. So Theorem 3.2 implies that $S^3_{G_i} \sim_{Y_{2n-1}} S^3$.

5 Trivalent diagrams and Goussarov–Vassiliev invariants for Brunnian links

In this section, we recall some results proved by Habiro and the author in a previous paper [14]. These, together with the two theorems shown in Section 4, will allow us to prove Theorem 1.3 in the next section.

5.1 Trivalent diagrams

A *trivalent diagram* is a finite graph with trivalent vertices, each vertex being equipped with a cyclic order on the three incident edges. The *degree* of a trivalent diagram is half the number of vertices.

For $k \geq 0$, let $\mathcal{A}_k(\emptyset)$ denote the $\mathbf{Z}$–module generated by trivalent diagrams of degree k, subject to the *AS and IHX relations*, see Figure 5.1.

Denote by $\mathcal{A}^c_k(\emptyset)$ the $\mathbf{Z}$–submodule of $\mathcal{A}_k(\emptyset)$ generated by *connected* trivalent diagrams.

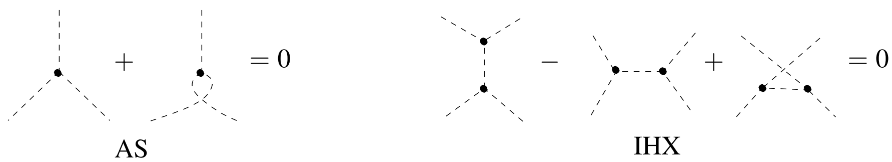

Figure 5.1: The AS and IHX relations

5.2 The Brunnian part of the Goussarov–Vassiliev filtration

Denote by $\mathbf{Z}\mathcal{L}(n)$ the free $\mathbf{Z}$–module generated by the set of isotopy classes of n–component links in S^3, and denote by $J_k(n)$ the $\mathbf{Z}$–submodule of $\mathbf{Z}\mathcal{L}(n)$ generated by elements of the form

$$[L; C_1, ..., C_p] := \sum_{S \subseteq \{C_1,...,C_p\}} (-1)^{|S|} L_S,$$

where L is an n–component link in S^3, and where the C_i ($1 \leq i \leq p$) are disjoint C_{k_i}–trees for L such that $k_1 + ... + k_p = k$. The sum runs over all the subsets S of $\{C_1, ..., C_p\}$ and $|S|$ denotes the cardinality of S. The descending filtration

$$\mathbf{Z}\mathcal{L}(n) = J_0(n) \supset J_1(n) \supset J_2(n) \supset ...$$

coincides with the *Goussarov–Vassiliev filtration* [12].

Denote by $\overline{J}_k(n)$ the graded quotient $J_k(n)/J_{k+1}(n)$.

Definition 5.1 The *Brunnian part* $\mathrm{Br}(\overline{J}_{2n}(n+1))$ of the $2n^{th}$ graded quotient $\overline{J}_{2n}(n+1)$ is the $\mathbf{Z}$–submodule generated by elements $[L-U]_{J_{2n+1}}$ where L is an $(n+1)$–component Brunnian link.

As outlined in [13, Section 7], $\mathrm{Br}(\overline{J}_{2n}(n+1))$ is spanned over $\mathbf{Z}$ by elements

$$\frac{1}{2}[U; T_\sigma \cup \tilde{T}_\sigma] \text{ and } [U; T_\sigma \cup \tilde{T}_{\sigma'}], \qquad \text{for } \sigma \neq \sigma' \in S_{n-1},$$

where, for all σ, σ' in the symmetric group S_{n-1}, T_σ is the simple linear C^a_n–tree for the $(n+1)$–component unlink U depicted in Figure 5.2, and $\tilde{T}_{\sigma'}$ is obtained from $T_{\sigma'}$ by a small isotopy so that it is disjoint from T_σ. (Here $\frac{1}{2}[U; T_\sigma \cup \tilde{T}_\sigma]$ means an element $x \in \mathrm{Br}(\overline{J}_{2n}(n+1))$such that $2x = [U; T_\sigma \cup \tilde{T}_\sigma]$. Existence of such an element is shown in [13].)

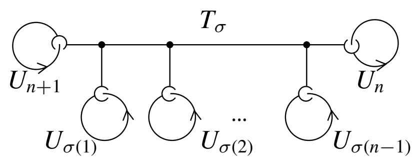

Figure 5.2: The simple linear C_n^a–tree T_σ

5.3 The map $h_n\colon \mathcal{A}^c_{n-1}(\emptyset) \to \mathrm{Br}(\overline{J}_{2n}(n+1))$

Connected trivalent diagrams allow us to describe the structure of $\mathrm{Br}(\overline{J}_{2n}(n+1))$. For $n \geq 2$, we have a map

$$h_n\colon\ \mathcal{A}^c_{n-1}(\emptyset) \longrightarrow \overline{J}_{2n}(n+1)$$

defined as follows. Given a connected trivalent diagram $\Gamma \in \mathcal{A}^c_{n-1}(\emptyset)$, insert $n+1$ ordered copies of S^1 in the edges of Γ, in an arbitrary way. The result is a strict unitrivalent graphs D_Γ of degree $2n$ on the disjoint union of $(n+1)$ copies of S^1 (see [1]). Next, 'realize' this unitrivalent graph by a graph clasper. Namely, replace each univalent vertex (resp. trivalent vertex, edge) of D_Γ with a disk-leaf (resp. node, edge), these various subsurfaces being connected as prescribed by the graph D_Γ. Denote by $C(D_\Gamma)$ the resulting graph clasper for the $(n+1)$–component unlink U. Then

$$h_n(\Gamma) := [U - U_{C(D_\Gamma)}]_{J_{2n+1}} \in \overline{J}_{2n}(n+1).$$

For $n \geq 3$, the image of h_n is the Brunnian part $\mathrm{Br}(\overline{J}_{2n}(n+1))$ of $\overline{J}_{2n}(n+1)$, and

$$h_n \otimes \mathbf{Q}\colon\ \mathcal{A}^c_{n-1}(\emptyset)\otimes\mathbf{Q} \longrightarrow \mathrm{Br}(\overline{J}_{2n}(n+1))\otimes\mathbf{Q}$$

is an isomorphism.

6 Finite type invariants of integral homology spheres

6.1 The Ohtsuki filtration for integral homology spheres

Let $\mathcal{M}$ denote the free $\mathbf{Z}$–module generated by the set of orientation-preserving homeomorphism classes of integral homology spheres. The definition of the Ohtsuki filtration uses algebraically split, unit-framed links. For the purpose of the present paper, it is however more convenient to use a definition using claspers, due to Goussarov and Habiro [4, 7, 12]. For $k \geq 0$, let $\mathcal{M}_k$ denote the $\mathbf{Z}$–submodule of $\mathcal{M}$ generated by elements of the form

$$[M; G_1, ..., G_p] := \sum_{S \subseteq \{G_1,...,G_p\}} (-1)^{|S|} M_S,$$

where M is an integral homology sphere, and where the G_i ($1 \le i \le p$) are disjoint Y_{k_i}–graphs in M such that $k_1 + ... + k_p = k$. The sum runs over all the subsets S of $\{G_1, ..., G_p\}$ and $|S|$ denotes the cardinality of S.

The descending filtration of $\mathbf{Z}$–submodules

$$\mathcal{M} = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset ...$$

is equal to the Ohtsuki filtration after re-indexing and tensoring by $\mathbf{Z}[1/2]$ [4, 7, 12]. Another alternative definition was previously given by Garoufalidis and Levine using 'blinks' [5].

6.2 The connected part of the Ohtsuki filtration

Let $\overline{\mathcal{M}}_{2k}$ denote the graded quotient $\mathcal{M}_{2k}/\mathcal{M}_{2k+1}$.

Definition 6.1 The *connected* part $\mathrm{Co}(\overline{\mathcal{M}}_{2k})$ of $\overline{\mathcal{M}}_{2k}$ is the $\mathbf{Z}$–submodule of $\overline{\mathcal{M}}_{2k}$ generated by elements $[S^3; G]_{\mathcal{M}_{2k+1}}$ where G is a Y_{2k}–graph (in particular, G is *connected*).

For $k \ge 1$, there is a well-defined *surgery map*

$$\varphi_k \colon\ \mathcal{A}_k(\emptyset) \longrightarrow \overline{\mathcal{M}}_{2k},$$

which maps each trivalent diagram $\Gamma = \Gamma_1 \cup ... \cup \Gamma_p$ to $[S^3; G_{\Gamma_1}, ..., G_{\Gamma_p}]$, where G_{Γ_i} is a connected clasper obtained by 'realizing' the diagram Γ_i in S^3 as depicted in Figure 6.1. The image $\varphi_k(\Gamma)$ of a degree k trivalent diagram Γ in $\overline{\mathcal{M}}_{2k}$ by φ_k does not depend on the embeddings G_{Γ_i} in S^3 ([12], see also [20, page 320]). Note that φ_k is a reconstruction, using claspers, of a map defined previously by Garoufalidis and Ohtsuki [6]. The homomorphism $\varphi_k \otimes \mathbf{Z}[1/2]$ is surjective, and it is an isomorphism

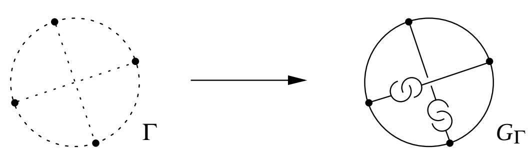

Figure 6.1: Realizing a trivalent diagram in S^3

when tensoring by $\mathbf{Q}$, with inverse given by the LMO invariant [15].

It can be easily checked using the arguments of [4] that $\varphi_k(\mathcal{A}_k^c(\emptyset)) = \mathrm{Co}(\overline{\mathcal{M}}_{2k})$. We thus have an isomorphism

$$\varphi_k \otimes \mathbf{Q} \colon\ \mathcal{A}_k^c(\emptyset) \otimes \mathbf{Q} \xrightarrow{\simeq} \mathrm{Co}(\overline{\mathcal{M}}_{2k}) \otimes \mathbf{Q}$$

induced by the surgery map φ_k.

6.3 The map $\alpha_k\colon\ \mathrm{Co}(\overline{\mathcal{M}}_{2k}) \longrightarrow \overline{\mathcal{S}}_{2k}$

Let $\mathcal{S}_k$ denote the set of integral homology spheres which are Y_k–equivalent to S^3, and denote by $\overline{\mathcal{S}}_k$ the quotient $\mathcal{S}_k/\sim_{Y_{k+1}}$. The connected sum induces an abelian group structure on $\overline{\mathcal{S}}_k$.

As recalled in the introduction, $\overline{\mathcal{S}}_{2k+1} = 0$ for all $k \geq 1$. $\overline{\mathcal{S}}_{2k}$ is generated by the elements S^3_G, where G is a Y_{2k}–graph in S^3 (for $k = 0$, we have $\overline{\mathcal{S}}_1 = \mathbf{Z}/2\mathbf{Z}$). There is a surjective homomorphism of abelian groups

$$\phi_k\colon\ \mathcal{A}^c_k(\emptyset) \longrightarrow \overline{\mathcal{S}}_{2k}$$

defined by $\phi_k(\Gamma) := [S^3_{G_\Gamma}]_{Y_{2k+1}}$, where G_Γ is a topological realization of the diagram Γ as in the definition of φ_k (see Figure 6.1). It is well known that ϕ_k is well-defined (see the proof of [20, Theorem E.20]).

The map ϕ_k is an isomorphism over the rationals. This is shown by using the primitive part of the LMO invariant z^{LMO} [20, pages 329–330].

Let

$$\alpha_k\colon\ \mathrm{Co}(\overline{\mathcal{M}}_{2k}) \longrightarrow \overline{\mathcal{S}}_{2k}$$

be the map defined by

$$\alpha_k([S^3;G]_{\mathcal{M}_{2k+1}}) = [S^3_G]_{Y_{2k+1}}.$$

The fact that α_k is well-defined follows from standard arguments of clasper theory, and is well known to experts.

The following is clear from the above definitions.

Lemma 6.2 *The following diagram commutes for all $k \geq 1$:*

$$\begin{tikzcd}
\mathcal{A}^c_k(\emptyset) \arrow[d, "\varphi_k"'] \arrow[dr, "\phi_k"] & \\
\mathrm{Co}(\overline{\mathcal{M}}_{2k}) \arrow[r, "\alpha_k"'] & \overline{\mathcal{S}}_{2k}
\end{tikzcd}$$

As a consequence, α_k is an isomorphism over the rationals.

6.4 The map λ_n

For simplicity, we work over the rationals in the rest of this section.

Let $n \geq 2$. Denote by B_{n+1} the set of isotopy classes of $(n+1)$–component Brunnian links in S^3. Define a linear map

$$\tilde{\lambda}_n \colon \mathbf{Q}B_{n+1} \to \mathcal{M}$$

by assigning each element $L \in B_{n+1}$ to $S^3_{(L,1)}$. Note that $\tilde{\lambda}_n$ is well-defined, as $S^3_{(L,+1)}$ is an integral homology sphere for all $L \in B_{n+1}$.

Denote by $\mathbf{I}$ the submodule of $\mathbf{Q}B_{n+1}$ generated by element $(L-L')$ such that $\tilde{\lambda}_n(L-L')$ is in $\mathcal{M}_{2n-1}$. The following follows immediately from [11] and Theorem 1.2.

Lemma 6.3 *Let L and L' be two link-homotopic (or C^a_{n+1}–equivalent) $(n+1)$–component Brunnian links. Then $L - L' \in \mathbf{I}$.*

Note that two link-homotopic $(n+1)$–component Brunnian links satisfy $L - L' \in J_{2n+1}(n+1)$ [13, Proposition 7.1]. We generalize Lemma 6.3 as follows.

Proposition 6.4 *Let L, L' be two $(n+1)$–component Brunnian links in S^3 such that $L - L' \in J_{2n+1}(n+1)$. Then $L - L' \in \mathbf{I}$.*

Proof Let B be an $(n+1)$–component Brunnian link in S^3. By [13, Section 5], we have $B \sim_{C^a_{n+1}} U_F$, where $F = T_1 \cup \ldots \cup T_m$ is a simple C^a_n–forest F for U in S^3 such that, for all $1 \leq i \leq p$, we have $T_i = T_{\sigma_i}$ for some $\sigma_i \in S_{n-1}$ (see Figure 5.2 for the definition of T_{σ_i}). By Lemma 6.3 we thus have

$$B \equiv U_F \bmod \mathbf{I}.$$

Observe that we have the equality

$$U_F = \sum_{F' \subseteq F} (-1)^{|F'|}[U; F'].$$

For all $F' \subseteq F$, denote by $G(F')$ the clasper obtained in S^3 by performing $(+1)$–framed surgery along U. As in Section 4.2, we have $\tilde{\lambda}_n(U_F) = S^3_{(U_F,+1)} \cong S^3_{G(F)}$. As each C^a_n–tree in F' is turned into a Y_{n-1}–tree of S^3 by this operation, we have $\tilde{\lambda}_n([U; F']) = [S^3; G(F')] \in \mathcal{M}_{(n-1).|F'|}$. In particular, $\tilde{\lambda}_n([U; F']) \in \mathcal{M}_{2n-2}$ for all F' with $|F'| \geq 3$. It follows that

$$B \equiv \sum_{F' \subseteq F \,/\, |F'| \leq 2} (-1)^{|F'|}[U; F'] \bmod \mathbf{I}.$$

By strictly the same arguments as in the proof of [13, Theorem 7.4], one can check that, for every $\sigma \in S_{n-1}$, $[U; T_\sigma] \equiv \frac{1}{2}[U; T_\sigma, \tilde{T}_\sigma]$ mod $\mathbf{I}$. It follows that

$$B \equiv U + \frac{1}{2} \sum_{1 \leq i \leq m} [U; T_{\sigma_i}, \tilde{T}_{\sigma_i}] + \sum_{1 \leq i \neq j \leq m} [U; T_{\sigma_i}, \tilde{T}_{\sigma_j}] \text{ mod } \mathbf{I}.$$

It follows that $L - L'$ is equal, modulo $\mathbf{I}$, to a linear combination of the form ($\alpha_{\sigma,\sigma'} \in \mathbf{Q}$)

$$\sum_{\sigma,\sigma' \in S_{n-1}} \alpha_{\sigma,\sigma'} [U; T_\sigma, \tilde{T}_{\sigma'}]. \tag{6–1}$$

By assumption, $L - L' \in J_{2n+1}(n+1)$. So (6–1) vanishes in $\mathrm{Br}(\overline{J}_{2n}(n+1))$, and is thus mapped by h_n^{-1} onto a linear combination of connected trivalent diagrams which vanishes in $\mathcal{A}^c_{n-1}(\emptyset)$. (6–1) is thus a linear combination of terms of the following two types.

(1) (AS) $[U; T_1, T_2] + [U; T_1', T_2']$, where $T_1 \cup T_2$ and $T_1' \cup T_2'$ differ by the cyclic order of the three edges attached to a node.

(2) (IHX) $[U; T_1, T_2] + [U; T_1', T_2'] + [U; T_1'', T_2'']$, where $T_1 \cup T_2$, $T_1' \cup T_2'$ and $T_1'' \cup T_2''$ are as claspers I, H and X of Figure 2.8.

Consider a term of type (1). By [4, Corollary 4.6], we have $\tilde{\lambda}_n([U; T_1, T_2] + [U; T_1', T_2']) \in \mathcal{M}_{2n-1}$. The same holds for terms of type (2) by [4, Theorem 4.11].

This completes the proof. □

By Theorem 1.1 and Proposition 6.4, we have a well-defined homomorphism

$$\lambda_n\colon\ \mathrm{Br}(\overline{J}_{2n}(n+1)) \to \overline{\mathcal{M}}_{2n-2}$$

by setting

$$\lambda_n([L - U]_{J_{2n+1}}) := [S^3 - S^3_{(L,+1)}]_{\mathcal{M}_{2n-1}}$$

6.5 Proof of Theorem 1.3

First, we show that λ_n actually takes its values in the connected part of the Ohtsuki filtration.

Recall from Section 5.2 that $\mathrm{Br}(\overline{J}_{2n}(n+1))$ is generated by elements $[U; T_\sigma \cup \tilde{T}_{\sigma'}]$, for $\sigma, \sigma' \in S_{n-1}$. Each component U_i of U intersects one disk-leaf f_i of T_σ and one disk-leaf f_i' of $T_{\sigma'}$. Denote by $G_{\sigma,\sigma'}$ the Y_{2n-2}–graph obtained from $T_\sigma \cup \tilde{T}_{\sigma'}$ by connecting, for each $1 \leq i \leq n+1$, the edges incident to f_i and f_i'.

Lemma 6.5 *For all $\sigma, \sigma' \in S_{n-1}$,*

$$\lambda_n([U; T_\sigma \cup \tilde{T}_{\sigma'}]) \equiv [S^3; G_{\sigma,\sigma'}] \text{ mod } \mathcal{M}_{2n-1}.$$

Consequently, we have

$$\lambda_n(\mathrm{Br}(\bar{J}_{2n}(n+1))) \subset \mathrm{Co}(\overline{\mathcal{M}}_{2n-2}).$$

Proof For any $\sigma, \sigma' \in S_{n-1}$, we have

$$\lambda_n([U; T_\sigma \cup \tilde{T}_{\sigma'}]) = -S^3_{G(T_\sigma \cup \tilde{T}_{\sigma'})} + S^3_{G(T_\sigma)} + S^3_{G(T_{\sigma'})} - S^3,$$

where, if F is a C_n^a–forest for U, $G(F)$ denotes the clasper obtained in S^3 by $(+1)$–framed surgery along U.

For all $\tau \in S_{n-1}$, $G(T_\tau)$ is a linear Y_{n-1}–tree whose leaves are all (-1)–special leaves. So by Theorem 3.2, there exists a union G_τ of Y_k–trees, $k \geq 2n-2$ such that $S^3_{G(T_\tau)} \cong S^3_{G_\tau}$.

On the other hand, $G(T_\sigma \cup \tilde{T}_{\sigma'})$ is obtained from $T_\sigma \cup \tilde{T}_{\sigma'}$ by replacing $f_i \cup f_i'$ by a pair of Hopf-linked (-1)–framed leaves (as illustrated in Figure 6.2), for $1 \leq i \leq n+1$. By Habiro's move 7 and 2, $G(T_\sigma \cup \tilde{T}_{\sigma'})$ is equivalent to the clasper C obtained by

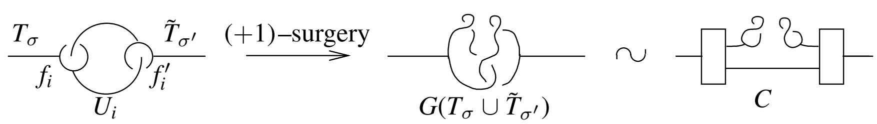

Figure 6.2: Performing $(+1)$–framed surgery along the unlink U

replacing each such pair of Hopf-linked leaves by two boxes as shown in Figure 6.2. By using the zip construction and Lemma 2.6, we obtain

$$S^3_C \cong S^3_{G(T_\sigma \cup \tilde{T}_{\sigma'})} \sim_{Y_{2n-1}} S^3_{G_{\sigma,\sigma'} \cup G(T_\sigma) \cup G(\tilde{T}_{\sigma'})}.$$

It follows that

$$\lambda_n([U; T_\sigma \cup \tilde{T}_{\sigma'}]) \equiv -S^3_{G_{\sigma,\sigma'} \cup G_\sigma \cup G_{\sigma'}} + S^3_{G_\sigma} + S^3_{G_{\sigma'}} - S^3 \text{ mod } \mathcal{M}_{2n-1}.$$

By using the equality $S^3_{G_{\sigma,\sigma'} \cup G_\sigma \cup G_{\sigma'}} = \sum_{G' \subseteq \{G_{\sigma,\sigma'}, G_\sigma, G_{\sigma'}\}} (-1)^{|G'|}[S^3; G']$, one can easily check that

$$S^3_{G_{\sigma,\sigma'} \cup G_\sigma \cup G_{\sigma'}} \equiv S^3_{G_{\sigma,\sigma'}} + S^3_{G_\sigma} + S^3_{G_{\sigma'}} - 2S^3 \text{ mod } \mathcal{M}_{2n-1}.$$

(here we use the fact that $G_{\sigma,\sigma'}$ and each connected component of G_σ and $G_{\sigma'}$ have degree $\geq 2n-2$). The result follows. $\square$

Clearly, the composite $\alpha_{n-1}\lambda_n$ is the map

$$\kappa_n\colon\ \mathrm{Br}(\bar{J}_{2n}(n+1)) \longrightarrow \overline{\mathcal{S}}_{2n-2}$$

announced in the statement of Theorem 1.3. By Lemma 6.2, it suffices to show that λ_n is an isomorphism to obtain the theorem. This is implied by the next lemma.

Lemma 6.6 *For $n \geq 3$, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{A}^c_{n-1}(\emptyset) \arrow[d, "h_n"'] \arrow[dr, "\varphi_n"] & \\
\mathrm{Br}(\bar{J}_{2n}(n+1)) \arrow[r, "\lambda_n"'] & \mathrm{Co}(\overline{\mathcal{M}}_{2n-2})
\end{tikzcd}$$

Proof As pointed out in [14, Section 3.5], one can easily check that $\mathcal{A}^c_{n-1}(\emptyset)$ is generated by the elements Γ_σ depicted in Figure 6.3, for all $\sigma \in S_{n-1}$.

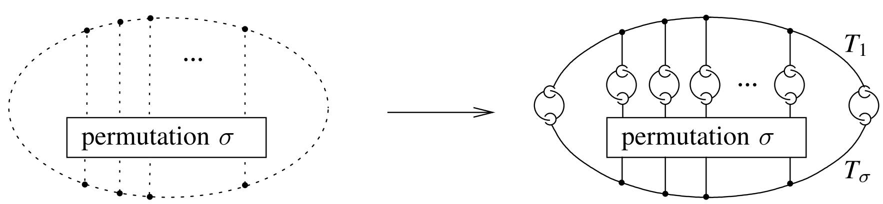

Figure 6.3: The connected trivalent diagram Γ_σ, and the two simple linear C^a_n–trees T_1 and T_σ

For such an element Γ_σ, a representative for $h_n(\Gamma_\sigma)$ is $[U; T_1 \cup T_\sigma]$, where T_1 and T_σ are two C^a_n–trees for U as represented in Figure 6.3. As seen in the proof of Lemma 6.5, $\lambda_n([U; T_1 \cup T_\sigma]) = [S^3; G_{1,\sigma}]_{\mathcal{M}_{2n-1}}$, where $G_{1,\sigma}$ is obtained by replacing each pair of disk-leaves intersecting the same component of U by an edge. Clearly, this Y_{2n-2}–graph satisfies $\varphi_n(\Gamma_\sigma) = [S^3; G_{1,\sigma}]_{\mathcal{M}_{2n-1}}$. □

The various results proved of this section can be summed up in the following commutative diagram ($n \geq 2$)

$$\begin{tikzcd}
 & \mathcal{A}^c_{n-1}(\emptyset) \arrow[dl, "h_n"'] \arrow[d, "\varphi_{n-1}"'] \arrow[dr, "\phi_{n-1}"] & \\
\mathrm{Br}(\bar{J}_{2n}(n+1)) \arrow[r, "\lambda_n"'] & \mathrm{Co}(\overline{\mathcal{M}}_{2n-2}) \arrow[r, "\alpha_{n-1}"'] & \overline{\mathcal{S}}_{2n-2}
\end{tikzcd}$$

where all arrows are isomorphism over $\mathbf{Q}$.

6.6 Brunnian links with vanishing Milnor invariants

In this last subsection, we can work over the integers.

Habegger and Orr also studied finite type invariants of integral homology spheres obtained by $(+1)$–framed surgery along links in S^3. In particular, [10, Theorem 2.1] deals with $(+1)$–framed surgery along l–component Brunnian links with vanishing Milnor invariants of length $\leq 2l-1$, and appears to have some similarities with our results.

Let $\mathrm{Br}^l(\bar{J}_k(n))$ denote the $\mathbf{Z}$–submodule of $\bar{J}_k(n)$ generated by elements $[L-U]_{J_{k+1}}$ where L is an n–component Brunnian link with vanishing Milnor invariants of length $\leq l$. Let $U_{(k)}$ denote the k–component unlink $U_1 \cup \cdots \cup U_k$ in S^3. Let

$$S_{n+1}\colon\ \mathrm{Br}(\bar{J}_{2n}(n+1)) \longrightarrow \mathbf{Z}\mathcal{L}(n)$$

be the map defined by

$$S_{n+1}([L-U_{(n+1)}]_{J_{2n+1}}) = s_{n+1}(L) - U_{(n)},$$

where $s_{n+1}(L)$ denotes the n–component link in S^3 obtained by $(+1)$–framed surgery along the $(n+1)^{th}$ component of L. In particular, $s_{n+1}(U_{(n+1)}) = U_{(n)}$.

We can show that, for $n \geq 3$,

(1) $S_{n+1}(\mathrm{Br}(\bar{J}_{2n}(n+1))) = \mathrm{Br}^{2n-1}(\bar{J}_{2n-1}(n))$

(2) $S_{n+1} \otimes \mathbf{Q}\colon\ \mathrm{Br}(\bar{J}_{2n}(n+1)) \otimes \mathbf{Q} \to \mathrm{Br}^{2n-1}(\bar{J}_{2n-1}(n)) \otimes \mathbf{Q}$ is an isomorphism.

The proof involves the same technique as in the preceding section, and makes use of Theorem 6.1 of [9].

7 The proof of Proposition 3.8

In this section, we give the proof of Proposition 3.8. For that purpose, it is convenient to state a few more technical lemmas on claspers.

Lemma 7.1 *The move of Figure 7.1 produces equivalent claspers.*

This is an easy consequence of [12, Proposition 2.7].

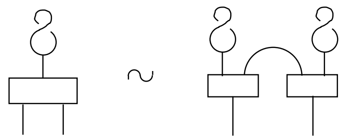

Figure 7.1

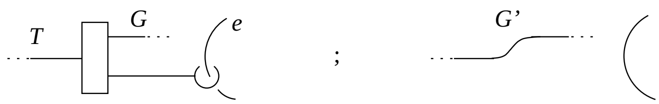

Figure 7.2

Lemma 7.2 *Let G be a clasper in a 3–manifold M containing a Y_k–subtree T, $k \geq 1$, such that a branch of T is incident to a box as shown in Figure 7.2. There, e is an edge of G which is not contained in T. Then*

$$M_G \sim_{Y_{k+1}} M_{G'},$$

where G' is the clasper depicted in the right-hand side of Figure 7.2.

The proof is omitted. It is straightforward, and uses Habiro's move 12 and a zip construction.

Lemma 7.3 *Let G be a clasper in a 3–manifold M such that a 3–ball B in M intersects G as depicted in Figure 7.3. There, the nodes n_1 and n_2 are both in a Y_k–subtree T, $k \geq 2$, and e is an edge of G which is not contained in T. Then*

$$M_G \sim_{Y_{k+1}} M_{G'},$$

where G' is identical to G outside of B, where it is as shown in Figure 7.3.

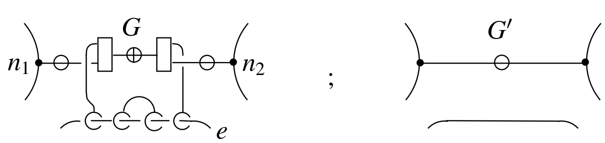

Figure 7.3

Proof By an isotopy, G is seen to be equivalent to the clasper G_1 represented in Figure 7.4. By applying the move of [12, Figure 38] to G_1, and then applying Habiro's

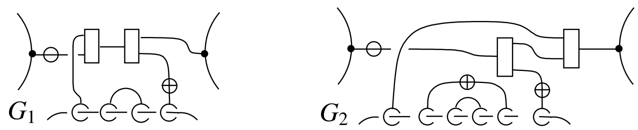

Figure 7.4

move 6 twice, we obtain the clasper $G_2 \sim G_1$ of Figure 7.4. Consider the two I–shaped claspers $I_1 \cup I_2$ of G_2 which appear in the figure. By Habiro's move 6 and 4, we have that $G_2 \sim G_2 \setminus (I_1 \cup I_2)$. The result then follows from Lemma 7.2. □

We can now prove Proposition 3.8.

Let G be a linear Y_n–tree in a 3–manifold M, $n \geq 2$, with $n+2$ (-1)–special leaves, and let N denote an s–regular neighborhood N of G. As noted previously, N is a 3–ball in M.

By $(n-1)$ applications of Lemma 2.3, G is equivalent to the clasper $\tilde{G}$ represented in Figure 7.5. The first step of this proof is to show the following.

Claim 7.4 We have

$$\tilde{G} \sim C,$$

in N, where C is the clasper containing a Y_{2n}–subtree represented in Figure 7.5.

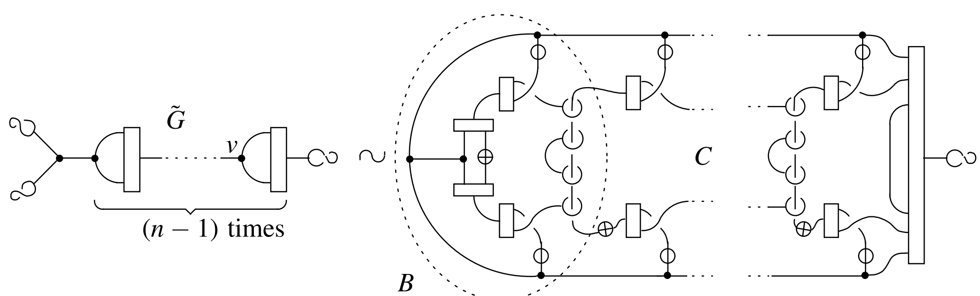

Figure 7.5

Proof Consider the box of $\tilde{G}$ which is connected to one (-1)–special leaf. This box is connected to a node v by two edges. By applying Lemma 2.4 at v, and Lemma 2.5, we obtain the clasper represented in Figure 7.6 (a). Then apply recursively Lemma 2.4

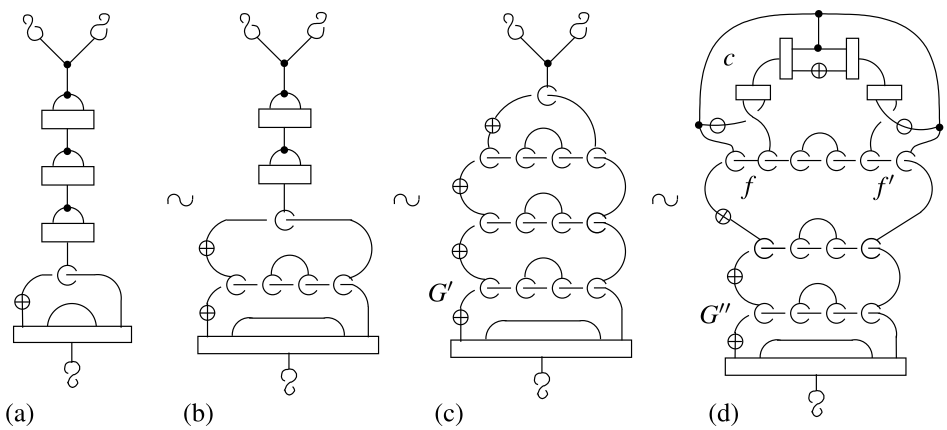

Figure 7.6: Here, for simplicity, we consider the case $n = 5$.

and Habiro's move 6, as shown in Figure 7.6 (b), until we obtain a clasper $G' \sim \tilde{G}$ with only one node connected to two (-1)–special leaves. See in Figure 7.6 (c). By applying the move of Figure 3.1 and Habiro's move 6, we have $G' \sim G''$, where G'' contains a component c with 4 nodes and with two leaves f and f' lacing an edge e – see Figure 7.6 (d).[7] We can apply Habiro's move 12 to these two leaves, and then Habiro's move 6 to create two new leaves lacing an edge. Apply recursively these two moves until no new leaf lacing an edge is created: the result is the desired clasper C which contains a Y_{2n}–subtree T, as represented in Figure 7.5. □

Consider in N a 3–ball B which intersects C as depicted. By several applications of the move of [12, Figure 38] and of Habiro's move 6, we obtain the clasper $G_1 \sim G$ which is identical to C outside B, where it is as shown in Figure 7.7. By Habiro's move 6 and 4, we can freely remove the pair of I–shaped claspers which appear in the figure (see the proof of Lemma 7.3). By further applying four times Lemma 7.2, we thus obtain the clasper G_2 of Figure 7.7, which satisfies $N_{G_2} \sim_{Y_{2n+1}} N_{G_1}$. By an isotopy, we can apply Habiro's move 12 to show that $N_{G_2} \sim N_{G_3}$, where G_3 is as shown in Figure 7.7. By using [20, page 398], we obtain $N_{G_3} \sim_{Y_{2n+1}} N_{G_4}$.[8]

Observe that G_4 satisfies the hypothesis of Lemma 7.3. Actually, we can apply

[7] Here we say that a leaf of a clasper G *laces* an edge if it forms an unknot which bounds a disk D with respect to which it is 0–framed, such that the interior of D intersects G once, transversally, at an edge.

[8] We use the up-most figure of [20, page 398]. The arguments given there are for graph claspers, but they can be used in our situation.

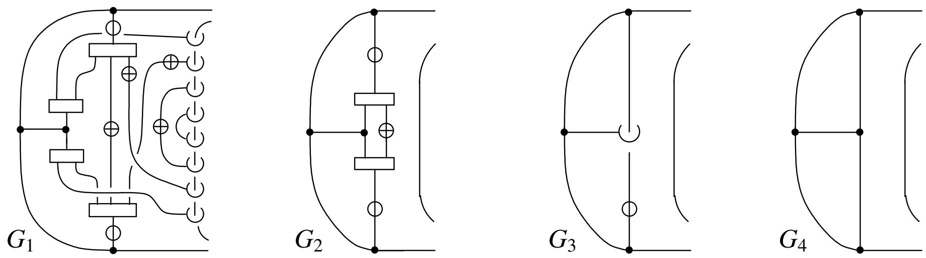

Figure 7.7: These four claspers are identical to C outside B.

Lemma 7.3 recursively $(n-3)$ times. By further applying, to the resulting clasper, strictly the same arguments as in the proof of Lemma 7.3, we obtain $N_{G_4} \sim_{Y_{2n+1}} N_{G_5}$, where G_5 is the clasper shown in Figure 7.8. It follows, by the zip construction and Lemma 2.6, that

$$N_{G_5} \sim_{Y_{2n+1}} N_{G_6 \cup G_7},$$

where G_6 and G_7 are two disjoint claspers in N as represented in Figure 7.8.

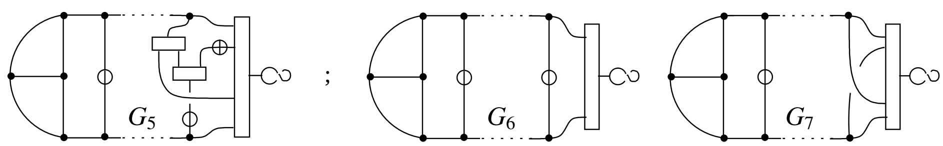

Figure 7.8

By Lemma 7.1 and Theorem 3.2 (for $l=1$), it is not hard to check that $N_{G_7} \sim_{Y_{2n+1}} N$ and that $N_{G_6} \sim_{Y_{2n+1}} N_{\Theta_n}$.

This concludes the proof of Proposition 3.8.

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